Lec 10: 09/24/2018

Bremsstrahlung (Cont'd):

The full expression in the non-relativistic limit, which uses full quantum mechanical treatment of the radiation process, is due to Bethe and Heitler. The key aspect of their result is that the electron cannot give up more than its total kinetic energy in the radiation process. Therefore no photons are radiated with frequencies higher than $\frac{(\gamma-1)m_e c^2}{\hbar}$. The Bethe-Heitler formula is:

$$\frac{d^2W}{d\omega\,dt} = \frac{8}{3}\frac{Z^2e^6}{c^3}\frac{\gamma v n}{m_e E_{kin}} \ln\left[\frac{1+\left(1-\frac{\varepsilon}{E_{kin}}\right)^{1/2}}{1-\left(1-\frac{\varepsilon}{E_{kin}}\right)^{1/2}}\right]$$

Here $E_{kin} = (\gamma-1)m_e c^2$ is the kinetic energy of the electron and $\varepsilon = \hbar\omega$ is the photon energy.

To find the total energy loss rate due to Bremsstrahlung, we should integrate $\frac{d^2W}{d\omega\,dt}$ over all frequencies. In practice, as pointed out above, this means integrating to a cut-off frequency $\omega_{max} = \frac{E}{\hbar}$. Then:

$$\frac{dE}{dt} = -\frac{8 Z^2 e^6 \gamma n v}{3c^3(\gamma-1)m_e^2 c^2}\int_0^{\omega_{max}} \ln\Lambda \, d\omega$$

Here Λ is given by the Bethe-Heitler formula on the previous page. It is convenient to write:

$$g(v,\omega) = \frac{\sqrt{3}}{\pi}\ln\Lambda$$

Here, $g(v,\omega)$ is a correction factor, which is called the Gaunt factor. In the non-relativistic limit, $\omega_{max} = \frac{m_e v^2}{2\hbar}$. This results in:

$$\frac{dE}{dt} \simeq -(\text{Const.})\frac{Z^2 n v}{m_e} \Rightarrow \frac{dE}{dt} \propto \frac{E^{\frac{1}{2}}}{m_e^{3/2}}$$

This implies that Bremsstrahlung loss is more efficient at higher energies. Also, as expected, energy loss via Bremsstrahlung is more efficient for lighter particles.

An important point to note is that $\frac{dE}{dt}$ is Lorentz invariant.

Therefore, the result obtained in the rest frame of the electron is also valid in the lab frame.

In the relativistic limit, the energy loss rate is given by the Bethe-Heitler formula derived from the full relativistic quantum treatment. It takes various effects like the electron shielding of the ions and electron-electron interactions into account. Here we just give the expression in the relativistic regime:

$$\frac{dE}{dt} = -\frac{8Z(Z+1.3)e^6 n}{3m_e^2 c\hbar} E\left[\ln\left(\frac{183}{Z^{1/3}}\right)+\frac{1}{8}\right]$$

We note that $\frac{dE}{dt} \propto -E$ in this case.

Thermal Bremsstrahlung:

In a practical set up, a flux of electrons with different velocities are present in a medium. Consider such a flux of electrons passing by ions with respective number densities n_e and n. The frequency-dependent emissivity of the medium is given by:

$$\frac{d^3W}{d\omega\, dt\, dv} = \frac{16\pi e^6}{3\sqrt{3}\, c^3 m_e^2 v} n_e n Z^2 g(v, \omega)$$

Here, we have assumed that electrons are in the non-relativistic limit. If electrons have a thermal distribution, then their number density follows the Maxwell-Boltzmann distribution:

$$n_e(v)\, dv = 4\pi n_e \left(\frac{m_e}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{m_e v^2}{2kT}\right) dv$$

Thermally averaged emissivity will then be:

$$\varepsilon_{ff}(\nu) \equiv \frac{d^3W}{d\nu\, dv\, dt} = \frac{32\pi^2 Z^2 e^6 n_e n}{3\sqrt{3}\, c^3 m_e^2} \mathrm{Int}(\nu, T)$$

Here:

$$\mathrm{Int}(\nu, T) \equiv \frac{\int_{v_{min}}^{\infty} g(\nu, v)\, v^2 \exp\left(-\frac{m_e v^2}{2kT}\right) dv}{\int_0^{\infty} v^2 \exp\left(-\frac{m_e v^2}{2kT}\right) dv}$$

We note the lower limit of the integral in the numerator. It is due to the fact that a minimum speed $v_{min} = \left(\frac{2h\nu}{m_e}\right)^{1/2}$

is required in order to emit a photon with frequency ν.

These integrals can be evaluated, which gives rise to the following result written in terms of a thermally-averaged Gaunt factor:

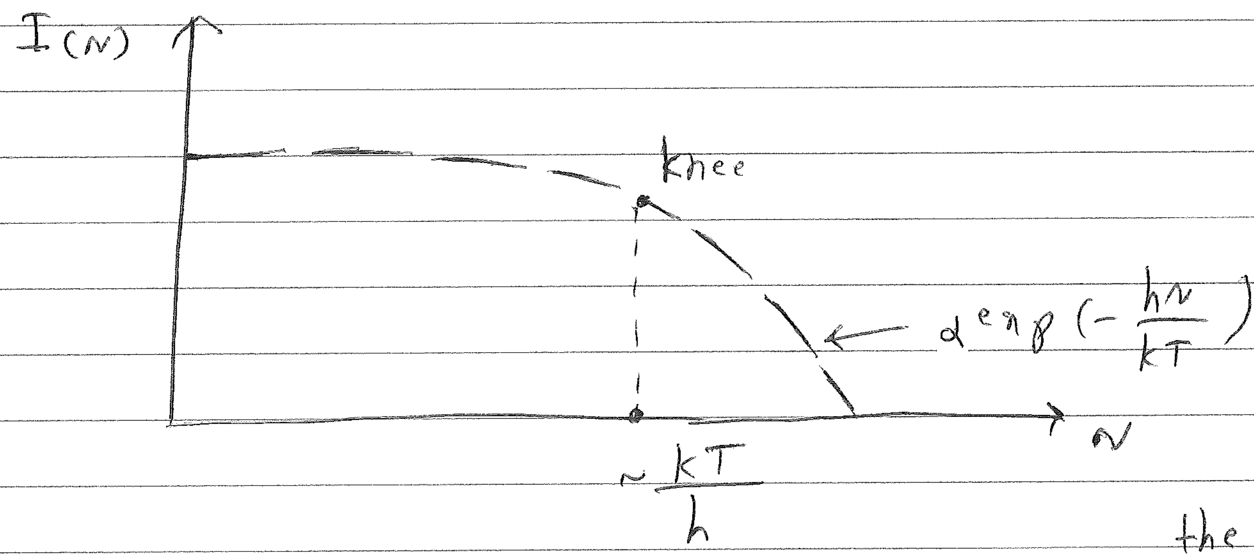
$$\varepsilon_{ff}(\nu) = 6.8 \times 10^{-38}\, n_e n\, Z^2\, T^{-\frac{1}{2}} \exp\left(-\frac{h\nu}{kT}\right) g(\nu, T) \;\; \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}}$$

At frequencies $\nu \ll \frac{kT}{h}$, $g(\nu, T)$ has only a logarithmic dependence on ν. Suitable form of $g(\nu, T)$ at X-ray frequencies (which are of interest to us) is:

$$g(\nu, T) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{kT}{h\nu}\right)$$

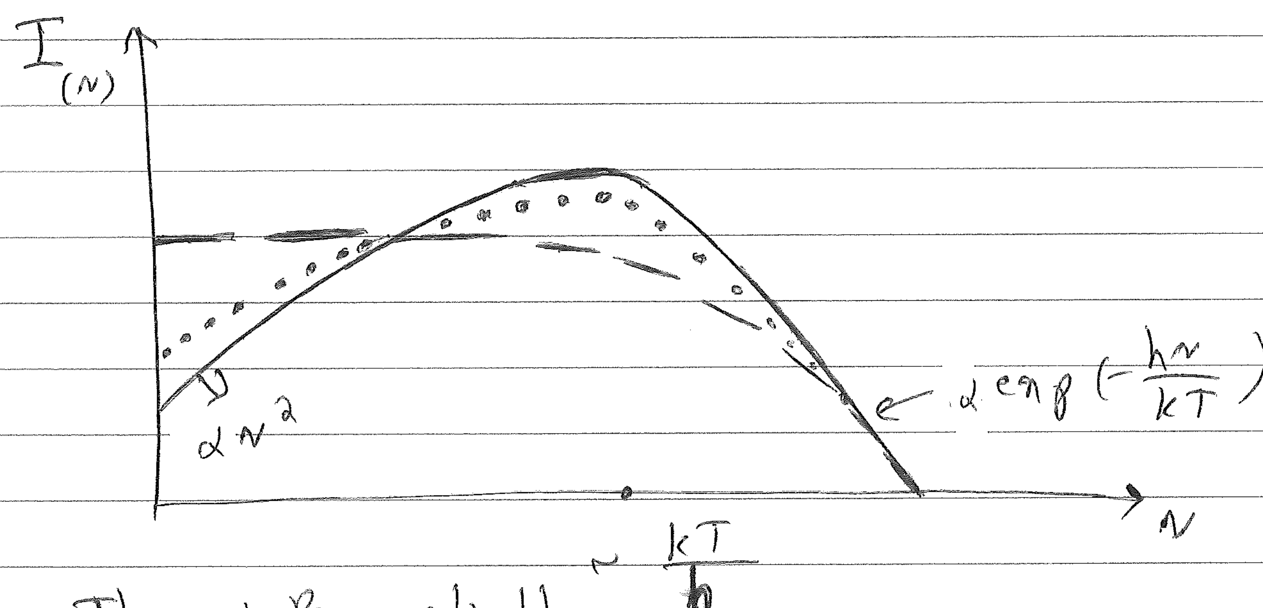
For the vast majority of astrophysical thermal plasmas, we have $1 \lesssim g(\nu, T) \lesssim 5$.

Due to the mild frequency dependence of $g(\nu, T)$ at low frequencies, the spectrum is flat for photon energies much smaller than the thermal energy kT. As we will see later, such a profile is unique among various emission mechanisms thereby providing a distinct observational signature. The spectrum turns over at $h\nu \sim kT$, and drops off exponentially at higher energies:



The existence of a "Bremsstrahlung knee" provides the means to measure the temperature of the emitting plasma. For example, the spectrum of the intergalactic gas in the Perseus cluster of galaxies, as observed in the X-ray waveband by the HEAO-A2 experiment, can be used to derive a temperature $T \doteq 7.5 \times 10^{7}$ K for the emitting gas. The interpretations of the diffuse X-ray emission from the clusters as the Bremsstrahlung of hot gas provides an important measure of the mass of the intergalactic gas in the clusters as well as providing an astrophysical tool for measuring the total mass of the clusters.

So far, we have assumed that the photons produced via Bremsstrahlung leave the system unaffected. In other words, we have considered the spectrum being that of the optically thin Bremsstrahlung spectrum. But this is not always the situation encountered in high energy astrophysics. The medium may not be thin enough for radiation to escape without further interaction with the plasma. If the mean free path of photons ℓ for scattering (or reabsorption) is large compared with the size R of the system, then we will have an optically thin emission. Since $\ell_s \frac{1}{n_e \sigma}$ (σ being the cross section for photon-electron interaction), this happens when $\tau \equiv n_e \sigma R \ll 1$, where τ is the optical depth. In the other limit, when $\tau \gg 1$, the photons undergo numerous interactions before leaving the system. Since particles in the medium have thermal distribution, the radiation will emerge with a black body spectrum in this case, which has a prominent bump at $\nu \sim \frac{kT}{h}$:



– – – : Thermal Bremsstrahlung
——— : Black body
. . . . . . : Intermediate